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A New Version of the Strang–Fix Conditions

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A new version of the Strang–Fix conditions is formulated and it is used to give a new proof for the characterization of the local approximation order of the spaces generated by a finite number of compactly supported basis functions and their shifts. © 1993 Academic Press, Inc.

Let Φ be a finite collection of compactly supported functions in $L_1(\mathbb{R}^s)$. We denote by span (Φ) the linear span of Φ, and by $S(\Phi)$ the linear space *spanned* by the funtions in Φ and their shifts. Here by *a shift* we mean a multi-integer translate. Let $\{e_1, ..., e_s\}$ be the standard basis for $\mathbb{R}^s$. Then the shift operator T_j $(j = 1, ..., s)$ is defined by $T_j f := f(\cdot - e_j)$ for all functions f defined on $\mathbb{R}^s$, and the difference operator ∇_j is defined to be $1 - T_j$, where 1 stands for the identity operator. Given a positive integer k, we say that the collection Φ satisfies *the Strang–Fix conditions of order k* if there is an element ψ of $S(\Phi)$ such that

$$\hat{\psi}(0) = 1$$

and

$$D^{\lambda}\hat{\psi}(2\pi\alpha) = 0$$

for all $\lambda \in \mathbb{N}^s$ with $|\lambda| < k$ and all $\alpha \in \mathbb{Z}^s \backslash \{0\}$, where $\hat{\psi}$ denotes the Fourier transform of ψ:

$$\hat{\psi}(\xi) := \int_{\mathbb{R}^s} \psi(x)\, e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^s.$$

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We have used the standard multi-index notation in the above. In particular, we used $\mathbb{N}$ to denote the set of nonnegative integers, and for $\lambda = (\lambda_1, ..., \lambda_s) \in \mathbb{N}^s$, the differential operator D^λ was defined to be $D_1^{\lambda_1} \cdots D_s^{\lambda_s}$, where D_j is the partial derivative operator with respect to the jth coordinate. Analogously, for $\alpha = (\alpha_1, ..., \alpha_s) \in \mathbb{Z}^s$ we denote by T^α the shift operator $T_1^{\alpha_1} \cdots T_s^{\alpha_s}$. The norm in $\mathbb{R}^s$ is chosen to be the l_1-norm:

$$|x| := |x_1| + \cdots + |x_s| \qquad \text{for} \quad x = (x_1, ..., x_s) \in \mathbb{R}^s.$$

If $\lambda \in \mathbb{N}^s$, then $|\lambda|$ is just *the length* of λ. Moreover, we denote by $\bar{S}(\Phi)$ the linear space *generated* by the functions in Φ and their shifts. In other words, $g \in \bar{S}(\Phi)$ if and only if g has a representation of the form

$$g = \sum_{\alpha \in \mathbb{Z}^s} \sum_{\phi \in \Phi} c_\phi(\alpha)\, \phi(\cdot - \alpha),$$

where for each $\phi \in \Phi$, c_ϕ is a complex sequence on $\mathbb{Z}^s$. Note that $g \in S(\Phi)$ if and only if in the above representation every c_ϕ is supported on a finite subset of $\mathbb{Z}^s$. Thus $\bar{S}(\Phi)$ is *the completion* (see [7, p. 38]) of $S(\Phi)$ under the topology of compact convergence. Given $h > 0$ and a space F of functions on $\mathbb{R}^s$, we denote by F_h the h-scaling of F:

$$F_h := \{f(\cdot/h) : f \in F\}.$$

The importance of the Strang–Fix conditions rests on their applications to the approximation problems related to the family $\{\bar{S}_h(\Phi) : h > 0\}$ of approximating spaces. Now it is well known that if Φ satisfies the Strang–Fix conditions of order k, then $\{\bar{S}_h(\Phi) : h > 0\}$ provides approximation of order k. More precisely, we have the following theorem.

THEOREM 1. *Let Φ be a finite collection of compactly supported functions in $L_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$. If Φ satisfies the Strang–Fix conditions of order k, then there is a constant C independent of h such that*

$$\operatorname{dist}_p(f, \bar{S}_h(\Phi)) \leqslant Ch^k\, |f|_{k,p} \qquad \textit{for all} \quad f \in W_p^k(\mathbb{R}^s),$$

where dist_p *indicates that the distance is measured in the L_p-norm, and W_p^k denotes the usual Sobolev space, while*

$$|f|_{k,p} := \sum_{|\lambda| = k} \|D^\lambda f\|_{L_p(\mathbb{R}^s)}.$$

Theorem 1 was first proved by Strang and Fix in [6] for the case $p = 2$. The general case $1 \leqslant p \leqslant \infty$ was discussed in [1] and [3]. A concrete scheme of L_p-approximation was given in [4].

The converse of Theorem 1 is not true in general. But de Boor and Jia succeeded in characterizing the local approximation order (see [1] for the definition) by proving the following theorem.

THEOREM 2. *For any p, $1 \leqslant p \leqslant \infty$, Φ provides local L_p-approximation of order k if and only if Φ satisfies the Strang–Fix conditions of order k.*

After a close look into the paper [1] we have found that the essence of the proof given there lies in the following fact.

THEOREM 3. *The following statements are equivalent.*

(i) *Φ satisfies the Strang–Fix conditions of order k.*

(ii) *There exists a sequence of elements $\psi_n \in S(\Phi)$ $(n = 1, 2, ...)$ such that as $n \to \infty$ the sequence*

$$\hat{\psi}_n(0) \to 1$$

and

$$D^\lambda \hat{\psi}_n(2\pi\alpha) \to 0$$

for all $\lambda \in \mathbb{N}^s$ with $|\lambda| < k$ and all $\alpha \in \mathbb{Z}^s \backslash \{0\}$.

Indeed, if Φ provides local approximation of order k, then as was proved in [1], for any $h > 0$ one can find $u_h \in S_h(\Phi)$ such that as $h \to 0$ the sequence $\hat{u}_h(0) \to 1$ and $D^\lambda \hat{u}_h(2\pi\alpha/h)/h^{k-1} \to 0$ for $|\lambda| < k$ and $\alpha \in \mathbb{Z}^s \backslash \{0\}$. Let $\psi_n := u_{1/n}(\cdot/n)/n^s$ $(n = 1, 2, ...)$. Then the sequence $(\psi_n)_{n=1,2,...}$ satisfies condition (ii) of Theorem 3. Thus Theorem 2 follows from Theorem 3 and Theorem 1.

In this note we shall give a new proof for Theorem 3 and discuss its possible extensions. For this purpose, we introduce a topological linear space V as follows. Let E be the set of those pairs $(\lambda, \alpha) \in \mathbb{N}^s \times \mathbb{Z}^s$ for which either $|\lambda| < k$ and $\alpha \in \mathbb{Z}^s \backslash \{0\}$, or $\lambda = 0$ and $\alpha = 0$. Consider the set V of all mappings from E to $\mathbb{C}$. It forms a linear space with respect to the usual pointwise addition and scalar multiplication. Furthermore, V is a Hausdorff topological linear space under the topology of pointwise convergence (see, e.g., [7, pp. 29–31]). Let L be the mapping from $S(\Phi)$ to V defined as follows. For any $\rho \in S(\Phi)$ we let $L\rho$ be the element in V given by

$$L\rho(\lambda, \alpha) = D^\lambda \hat{\rho}(2\pi\alpha), \qquad (\lambda, \alpha) \in E.$$

Then L is a linear mapping. Let b be the element of V given by

$$b(\lambda, \alpha) = \begin{cases} 1, & \text{if } (\lambda, \alpha) = (0, 0); \\ 0, & \text{otherwise.} \end{cases}$$

Then condition (i) of Theorem 3 says that $b \in L(S(\Phi))$, and condition (ii) of Theorem 3 is equivalent to having b in the closure of $L(S(\Phi))$. Thus Theorem 3 will be proved if we can show that $L(S(\Phi))$ is closed. For this it suffices to show that $L(S(\Phi))$ is finite dimensional, because any finite dimensional subspace of a Hausdorff topological linear space is closed (see, e.g., [7, Coro. 9.2]).

In order to prove that $L(S(\Phi))$ is finite dimensional, we first investigate the kernel space N of L:

$$N := \{ \rho \in S(\Phi) : L\rho = 0 \}.$$

We observe that for any $\rho \in S(\Phi)$, $\nabla_j^k \rho \in N$ for $j = 1, ..., s$. Indeed,

$$(\nabla_j^k \rho)^\wedge = g_j^k \hat{\rho},$$

where $g_j(\xi) := 1 - e^{i\xi_j}$ for $\xi = (\xi_1, ..., \xi_s) \in \mathbb{R}^s$. Clearly, $D^\lambda g_j^k(2\pi\alpha) = 0$ for all $|\lambda| < k$ and $\alpha \in \mathbb{Z}^s$. Therefore, with the aid of the Leibniz formula for differentiation, we conclude that

$$D^\lambda((\nabla_j^k \rho)^\wedge)(2\pi\alpha) = 0$$

for all $|\lambda| < k$ and $\alpha \in \mathbb{Z}^s$. This shows that $\nabla_j^k \rho \in N$.

Let

$$Q := \sum_{|\alpha| \leqslant ks} T^\alpha(\operatorname{span}(\Phi)).$$

We claim that for any $\alpha \in \mathbb{Z}^s$ and $\rho \in \operatorname{span}(\Phi)$, $T^\alpha \rho \in Q + N$. In other words, $S(\Phi) = Q + N$. This claim can be verified by induction on $|\alpha|$. Let $\rho \in \operatorname{span}(\Phi)$. If $|\alpha| \leqslant ks$, then $T^\alpha \rho \in Q$. Let $\alpha \in \mathbb{Z}^s$ be such that $|\alpha| > ks$ and suppose that $T^\beta \rho \in Q + N$ for all $\beta \in \mathbb{Z}^s$ with $|\beta| < |\alpha|$. Since $|\alpha| > ks$, some component of α, say α_j, has absolute value greater than k. Then either $\alpha_j < -k$ or $\alpha_j > k$. In the former case, we observe that $T^\alpha \rho$ is a linear combination of $\nabla_j^k(T^\alpha \rho)$ and $T_j^r(T^\alpha \rho)$, $r = 1, ..., k$. But $|\alpha + re_j| < |\alpha|$ for $r = 1, ..., k$ in this case. Hence by the induction hypothesis, we have $T_j^r(T^\alpha \rho) \in Q + N$ for $r = 1, ..., k$. This shows that $T^\alpha \rho \in Q + N$. The same argument is valid for the case $\alpha_j > k$. Thus we have proved that $S(\Phi) = Q + N$, from which we conclude immediately that $L(S(\Phi)) = L(Q)$ is finite dimensional. This completes the proof of Theorem 3.

Theorem 3 can be extended in various ways. First, the functions in Φ are not necessarily compactly supported. We only need to assume that any function ρ in Φ is in $L_1(\mathbb{R}^s)$, and the Fourier transform $\hat{\rho}$ is in C^{k-1} in some neighborhood of $2\pi\alpha$ for every $\alpha \in \mathbb{Z}^s$. Second, E could be any nonempty subset of $\{\lambda \in \mathbb{N}^s : |\lambda| < k\} \times \mathbb{Z}^s$. Again, we denote by V the topological linear space consisting of all mappings from E to $\mathbb{C}$, and let L

be the linear mapping from $S(\Phi)$ to V given by $L\rho(\lambda, \alpha) = D^{\lambda}\hat{\rho}(2\pi\alpha)$ for $\rho \in S(\Phi)$ and $(\lambda, \alpha) \in E$. Then we have the following result.

THEOREM 4. *There is a finite dimensional subspace Q of $S(\Phi)$ such that $S(\Phi) = Q + N$, where N is the kernel space of L. Consequently, the image of $S(\Phi)$ under the mapping L is a finite dimensional subspace of V, and hence is closed.*

Theorem 4 can be applied to approximation by shifts of functions having global support (see [4] and [5]) and to the Strang–Fix conditions associated to a lower set in $\mathbb{N}^s$ as studied in [2].

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